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GROUP ANALYSIS OF THE GENERALIZED KORTEWEG—DeVRIES—BURGERS EQUATIONS*

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Methods of group theory /1/ are used for analyzing the generalized Korteweg—DeVries—Burgers equation which defines the effects of nonlinearity, dispersion, and dissipation in many problems of continuous medium mechanics. Solutions belonging to the class of invariant solutions are obtained. The possibility of using these solutions in problems with initial conditions is considered.

1. Let us consider the generalized Korteweg—DeVries—Burgers equation of the form

$$\frac{\partial u}{\partial t}+\frac{j}{2}\frac{u}{t}+\gamma u^m\frac{\partial u}{\partial x}-\mu\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^3 u}{\partial x^3}=0 \tag{1.1}$$

where $j=0,1,2$ for the cases of plane, cylindrical, and spherical symmetry, respectively, m,γ,μ,β are some constants, $u(x,t)$ is the sought function, x is a space coordinate, and t is the second independent variable (the time).

Particular cases of Eq.(1.1) are encountered in investigation of waves in cold plasma /2—5/, in magnetohydrodynamics /6/, in the theory of motion of a fluid with bubbles /7/, and in problems of nonlinear acoustics /8/. We obtain invariant solutions of this equation using the method of group analysis /1/.

Besides determining the algebra of operators admitted by Eq. (1.1), optimal systems of subalgebras were sought and, also, the respective one-parameter subgroups and invariant solutions were analyzed. This analysis enabled us to obtain all invariant solutions constructed on dissimilar subgroups. The optimal system of subalgebras is given below, as an example, for $j=2$, $m=1$, and $\mu=0$. In remaining cases only the more interesting in the authors' opinion, invariant solutions are presented.

First, let us consider the case of plane symmetry $(j=0)$. Then when $\mu\neq 0,\beta\neq 0$ we have: when $m=1$ Eq.(1.1) admits the following transform operators:

$$X_1=\frac{\partial}{\partial x};\quad X_2=\frac{\partial}{\partial t},\quad X_3=\gamma t\frac{\partial}{\partial x}+\frac{\partial}{\partial u} \tag{1.2}$$

when $m=0$ Eq.(1.1) is linear, the operator space is infinite dimensional, but the meaningful part is contained in the space of operators L_4

$$X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t},\quad X_3=u\frac{\partial}{\partial u},\quad X_4=\left[x+2\left(\gamma-\frac{\mu^2}{3\beta}\right)t\right]\frac{\partial}{\partial x}+3t\frac{\partial}{\partial t}+\frac{\mu(x+\gamma t)}{3\beta}u\frac{\partial}{\partial u} \tag{1.3}$$

when $m\neq 0, m\neq 1$ only two operators remain

$$X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t} \tag{1.4}$$

Each of the operators (1.2)—(1.4) or their linear combination with constant coefficients generates group G_1 admitted by Eq.(1.1). For function $F(x,t)\not\equiv \text{const}$ to be an invariant of group G_1 with operator X it is necessary and sufficient that the equality $XF(x,t)=0$ is satisfied. For instance, for group G_1 defined by operator $X=\alpha X_1+X_2$, $\alpha=\text{const}$, $I=x-\alpha t$ is the invariant, and $u(x,t)=U(x-\alpha t)$ is the invariant solution which is called stationary.

2. In what follows we assume $m=1$. When $\gamma=1,\mu=0$, (1.1) is a Korteweg—DeVries equation.

In the plane case it admits the following transform operators:

$$X_1=\frac{\partial}{\partial x},\quad X_2=\frac{\partial}{\partial t},\quad X_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\quad X_4=x\frac{\partial}{\partial x}+3t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u}$$

to which correspond the invariant solutions

$$u_1=U(t),\quad u_2=U(x),\quad u_3=\frac{x}{t}+U(t),\quad u_4=\frac{x}{t}U(\lambda),\quad \lambda=\frac{x}{t^{1/3}}$$

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The last solution is also called the self-similar solution of the Korteweg—DeVries equation /2/. In addition to the indicated stationary solution $u = U(x - \alpha t)$ investigated in detail in /5/, we shall consider group G_1 defined by the operator $X = \alpha X_2 + X_3$. Here, $I_1 = x - t^2/(2\alpha)$ and $I_2 = \alpha u - t$ are the two independent invariants, and the invariant solution is of the form

$$u(x, t) = \frac{t + U(\lambda)}{\alpha}, \quad \lambda = x - \frac{t^2}{2\alpha} \tag{2.1}$$

The substitution of (2.1) into the Korteweg—DeVries equation yields

$$1 + \frac{U}{\alpha} U' + \beta U''' = 0$$

from which after integration and substitution

$$y(z) = -\beta (12\alpha\beta^2)^{-3/5} U(\lambda), \quad z = (12\alpha\beta^2)^{-1/5}\lambda$$

we obtain the equation $y'' = 6y^2 + z + C$ which yields known transcendental functions of Painlevé. We also point out the invariant solution

$$u(x, t) = \frac{\alpha}{2} + \frac{(x/t)\, U(\lambda)}{1 + \alpha t^{2/3}/(2\lambda)}, \quad \lambda = x t^{-1/3} - \frac{\alpha}{2} t^{2/3}$$

which is the same as the self-similar solution for $\alpha = 0$.

In the cylindrical case the Korteweg—DeVries equation admits the following transform operators:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x\sqrt{t}\,\frac{\partial}{\partial x} + 4t\sqrt{t}\,\frac{\partial}{\partial t} + \left(\frac{x}{\sqrt{t}} - 4u\sqrt{t}\right)\frac{\partial}{\partial u}, \quad X_3 = 2\sqrt{t}\,\frac{\partial}{\partial x} + \frac{1}{\sqrt{t}}\frac{\partial}{\partial u}, \quad X_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$$

and in the spherical case

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \ln t \frac{\partial}{\partial x} + \frac{1}{t}\frac{\partial}{\partial u}, \quad X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$$

The invariant solutions corresponding to groups with extension operators $X_4\ (j = 1)$ and $X_3\ (j = 2)$ are, obviously, self-similar solutions of the Korteweg—DeVries equation. Moreover, when $j = 1$ we have invariant solutions of the form

$$u(x, t) = \frac{x}{2t} + U(t), \quad u(x, t) = \frac{x/2 + U(\lambda)}{t}, \quad \lambda = x t^{-1/2}$$

Let us investigate in greater detail the case of $j = 2$. Using the methods in /1/ it is possible to show that in this case the optimal system of subalgebras is of the form

$$X_1,\ X_2,\ X_3,\ X_1 + X_2,\ X_1 + X_3,\ X_2 + X_3$$

The respective invariant solutions are

$$u_1 = U(t), \quad u_2 = \frac{x}{t \ln t} + U(t), \quad u_3 = \frac{x}{t} U(\lambda), \quad \lambda = x t^{-1/3}$$

$$u_4 = \frac{x}{t(\ln t + 1)} + U(t), \quad u_5 = t^{-2/3} U(\lambda), \quad \lambda = (x + 1)\, t^{-1/3}$$

$$u_6 = t^{-2/3} U(\lambda) - 1/t, \quad \lambda = (x + \ln t + 3)\, t^{-1/3}$$

There are no other solutions on dissimilar subgroups.

If $\gamma = 0$ with $\mu = 0, \beta \neq 0$, we have the linearized Korteweg—DeVries equation which admits an infinite dimension space of transform operators but the meaningful part is contained in the subspace of operators L_6

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t} - j\frac{u}{2t}\frac{\partial}{\partial u}, \quad X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t}, \quad X_4 = u\frac{\partial}{\partial u}, \quad X_5 = x t^{-j/2}\frac{\partial}{\partial u}, \quad X_6 = t^{-j/2}\frac{\partial}{\partial u}$$

Derivation of all possible invariant solutions is now not difficult.

3. When $\gamma = 1, \mu \neq 0, \beta = 0$, Eq. (1.1) is the known Burgers equation. The total system of admissible operators is

$$j = 0: \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad X_3 = \frac{\partial}{\partial t}$$

$$X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_5 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - ut)\frac{\partial}{\partial u}$$

$$j = 1: \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$$

$$X_3 = 2\sqrt{t}\,\frac{\partial}{\partial x} + \frac{1}{\sqrt{t}}\frac{\partial}{\partial u}$$

$$j = 2: \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$$

$$X_3 = \ln t\frac{\partial}{\partial x} + \frac{1}{t}\frac{\partial}{\partial u}$$

Invariant solutions in the case of $j=0$ were also investigated by Katkov (see /1/). When the operators are known it is possible to obtain a complete system of invariant solutions, which is not presented here. We would only point out that the substitution (2.1) for $j=0$ reduces the Burgers equation to that of Riccati. By the substitution

$$u(x,t) = -2\mu \frac{\partial \ln \theta(x,t)}{\partial x} \tag{3.1}$$

the Burgers equation reduces for $j=0$ to the thermal conductivity equation $\theta_t = \mu\theta_{xx}$ /9/. A large number of respective solutions is given in /10/. Although transform (3.1) is not invariant, there are many invariant solutions among those listed in /10/. For instance

$$u(x,t) = \lambda + \frac{U(\lambda)}{t}, \quad \lambda = \frac{x}{t} \tag{3.2}$$

is an invariant solution that corresponds to the group with operators X_5. Solutions (2.1)—(2.5) and (3.5) in /10/ are of this type, but not all solutions of type (3.2) are represented there. For instance, from among the set of invariant solutions

$$u(x,t) = \frac{x}{t} + \frac{2\mu}{t}\frac{C_1}{C_2 - C_1\frac{x}{t}}, \quad C_1, C_2 = \text{const}$$

that correspond to solutions

$$\theta(x,t) = \left(C_2 - C_1\frac{x}{t}\right)t^{-1/2}\exp\left(-\frac{x^2}{4\mu t}\right)$$

only two solutions that obtain when $C_1 = 0$ and $C_2 = 0$ appear in /10/.

4. Let us consider the problem of point explosion for the Burgers equation when $j=0$. We use the self-similar solution based on the extension operator X_2. The ordinary second order differential equation derived from the Burgers equation is integrable, and the solution of this problem is of the form

$$u(x,t) = \sqrt{\frac{\mu}{\pi t}}E_0\exp(-y^2)\Bigg/\left[2\mu + \frac{E_0}{\sqrt{\pi}}\int_y^\infty \exp(-y^2)\,dy\right], \quad y = \frac{x}{\sqrt{4\mu t}}, \quad E_0 = \text{const} \tag{4.1}$$

A direct substitution will show that (4.1) is a solution of the Burgers equation. It is evident that

$$\lim_{t\to+0} u(x,t) = E_0\delta(x), \quad \int_{-\infty}^{\infty} u(x,0)\,dx = E_0$$

Q.E.D. /11/.

Solution (4.1) can be also obtained using the substitution (3.1) /12/.

The results of this investigation of the Korteweg—DeVries—Burgers equation may prove useful for deriving new exact solutions, in the investigation of nonlinear wave properties, and in other physical problems.

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